

Intersection numbers from the antisymmetric Gaussian matrix model

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ABSTRACT: The matrix model of topological field theory for the moduli space of p -th spin curves is extended to the case of the Lie algebra of the orthogonal group. We derive a new duality relation for the expectation values of characteristic polynomials in the antisymmetric Gaussian matrix model with an external matrix source. The intersection numbers for non-orientable surfaces of spin curves with k marked points are obtained from the Fourier transform of the k -point correlation functions at the critical point where the gap is closing.

KEYWORDS: Matrix Models, Topological Field Theories.

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1. Introduction

For Riemann surfaces, it is known from Witten's conjectures [2] and Kontsevich's derivation [1] that the intersection numbers of the moduli space of curves with marked points may be obtained from an Airy matrix model. Furthermore, higher Airy matrix models have been shown to give the intersection numbers of the moduli space for p-spin curves [3].

Recently, a duality relation has been applied to this problem [4–6]. The derivation relied on a duality between the higher p-th Airy matrix models and Gaussian matrix models in an external matrix source at critical values of this source [7, 8] ; the intersection numbers are then easily obtained from this dual model. When p=2, the model reduces to Kontsevich's model ; its dual connects to the behavior of correlation functions near the edge of the semi-circle spectrum [9, 4].

The moduli space of p-th spin curves is described by random Hermitian matrices, the Lie algebra of the unitary group $U(N)$. It is of interest to extend this moduli space of spin curves for non-orientable surfaces, both in the fields of open string theory and of quantum chaos.

For obtaining non-orientable surfaces from standard loop-expansions of matrix models, a first possibility would be to use real symmetric matrices. The Euler characteristics of the moduli spaces of real algebraic curves with marked points may be obtained from the real

symmetric matrix model [10]. However, for the intersection numbers of the moduli space of curves with marked points, this real symmetric matrix model remains difficult to solve when one extends Kontsevich Airy matrix model to non-orientable surfaces [11].

In this article we have chosen, instead of real symmetric matrices, to consider real antisymmetric matrices, the Lie algebra of the $SO(N)$ group (we assume that N is an even integer). Let us note that the Gaussian random matrix model of the classical groups $O(N)$ and $Sp(N)$ appeared earlier in the literature in the studies of the moments of the \mathcal{L} -functions [12] and in the study of the spectrum of excitations inside superconducting vortices [13].

For the $O(N)$ matrices, there is a Harish Chandra formula for the integrals over the orthogonal group. Thanks to this integral formula, which is similar to the unitary case, generating functions of the intersection numbers for non-orientable surfaces become calculable. We shall discuss a duality relation for the $O(N)$ case, which is surprisingly similar to the $U(N)$ case ; then we compute explicit expansions for the Fourier transforms of the correlation functions of the dual models, and obtain the intersection numbers. This study may shed some light on the moduli space of curves on non-orientable surfaces.

2. Duality relation

Let us first state the basic duality relation which will be used in this article.

Theorem 1.

$$\left\langle \prod_{\alpha=1}^k \det(\lambda_{\alpha} \cdot I - X) \right\rangle_A = \left\langle \prod_{n=1}^N \det(a_n \cdot I - Y) \right\rangle_{\Lambda} \tag{2.1}$$

where X is $2N \times 2N$ real antisymmetric matrix ($X^t = -X$) and Y is $2k \times 2k$ real antisymmetric matrix ; the eigenvalues of X and Y are thus pure imaginary. A is also a $2N \times 2N$ antisymmetric matrix, and it couples to X as an external matrix source. The matrix Λ is $2k \times 2k$ antisymmetric matrix, coupled to Y . We assume, without loss of generality, that A and Λ have the canonical form:

$$A = \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots \\ -a_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & -a_2 & 0 & 0 \\ \dots & & & & \end{pmatrix}, \tag{2.2}$$

i.e.

$$A = a_1 v \oplus \dots \oplus a_N v, \quad v = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.3}$$

Λ is expressed also as

$$\Lambda = \lambda_1 v \oplus \dots \oplus \lambda_k v. \tag{2.4}$$

The characteristic polynomial $\det(\lambda \cdot \mathbf{I} - X)$ has the $2N$ roots $(\pm i\lambda_1, \dots, \pm i\lambda_n)$. The Gaussian averages in (2.1) are defined as

$$\begin{aligned} \langle \cdots \rangle_A &= \frac{1}{Z_A} \int dX e^{\frac{1}{2} \text{tr} X^2 + \text{tr} X A} \\ \langle \cdots \rangle_\Lambda &= \frac{1}{Z_\Lambda} \int dY e^{\frac{1}{2} \text{tr} Y^2 + \text{tr} Y \Lambda} \end{aligned} \quad (2.5)$$

in which X is a $2N \times 2N$ real antisymmetric matrix, and Y a $2k \times 2k$ real antisymmetric matrix; the coefficients Z_A and Z_Λ are such that the expectation values of one is equal to one. The derivation of Theorem 1 relies on a representation of the characteristic polynomials in terms of integrals over Grassmann variables, as for the $U(N)$ or $U(N)/O(N)$ [12, 14, 15] cases. Given the complexity of the intermediate steps of the derivation for the $O(N)$ case, the simplicity of the result is striking. The derivation is given in appendix A.

3. Higher Airy matrix models

From the theorem 1, we can obtain easily the higher Airy matrix models. We first consider the simple case in which the source A is a multiple of identity: $a_n = a$. Then we write

$$\begin{aligned} \left\langle \prod_{i=1}^N \det(a_n \cdot \mathbf{I} - Y) \right\rangle_\Lambda &= \langle [\det(a \cdot \mathbf{I} - Y)]^N \rangle_\Lambda \\ &= \frac{1}{Z_\Lambda} \int dY e^{N \text{tr} \log(a \cdot \mathbf{I} - Y) + \frac{1}{2} \text{tr} Y^2 + \text{tr} Y \Lambda} \end{aligned} \quad (3.1)$$

Expanding the logarithmic term, and noting that the traces of odd powers of Y vanish since Y is antisymmetric, we obtain

$$\langle [\det(a - Y)]^N \rangle_\Lambda = \frac{1}{Z_\Lambda} \int dY e^{2kN \log a - (\frac{N}{2a^2} - \frac{1}{2}) \text{tr} Y^2 - \frac{N}{4a^4} \text{tr} Y^4 + \dots + \text{tr} Y \Lambda} \quad (3.2)$$

Choosing $a^2 = N$, the coefficient of $\text{tr} Y^2$ vanishes. Then one rescales $Y \rightarrow N^{\frac{1}{4}} Y$, and $\Lambda \rightarrow N^{-\frac{1}{4}} \Lambda$. After these rescalings, we obtain in the large N limit, the higher Airy matrix model,

$$Z = \int dY e^{-\frac{1}{4} \text{tr} Y^4 + \text{tr} Y \Lambda} \quad (3.3)$$

Note that higher powers of Y^{2n} disappear in this scaling limit since they are given by

$$\frac{1}{nN^{n-1}} \cdot N^{\frac{2n}{4}} \text{tr} Y^{2n} \sim N^{-\frac{n}{2}+1} \text{tr} Y^{2n} \quad (3.4)$$

which vanish in the large- N limit for $n > 2$.

By appropriate tuning of the a_n 's, and corresponding rescaling of Y and Λ , one may generate similarly higher models of type

$$Z = \int dY e^{-\frac{1}{p+1} \text{tr} Y^{p+1} + \text{tr} Y \Lambda} \quad (3.5)$$

where p is an odd integer. These models are similar to the generalized Kontsevich model in the unitary case, which gives the intersection numbers of the moduli spaces of p -th spin curves. However, the matrix Y being real and antisymmetric, the partition function Z is very different from the unitary case and non-orientable surfaces lead to different intersection numbers.

4. Expansion in inverse powers of lambda

The free energy $F = \log Z$ can be expanded in powers of $\text{tr} \Lambda^{-m}$ as in the unitary case. This is done through the Harish Chandra formula [16] for the integration over the orthogonal group $g = \text{SO}(2N)$. We may take Y and Λ in canonical form (2.1) without loss of generality: then the Harish Chandra integral reads [16]

Theorem 2. (Harish Chandra)

$$\int_{\text{SO}(2N)} dg e^{\text{tr}(gYg^{-1}\Lambda)} = C \frac{\sum_{w \in W} (\det w) \exp \left[2 \sum_{j=1}^N w(y_j) \lambda_j \right]}{\prod_{1 \leq j < k \leq N} (y_j^2 - y_k^2)(\lambda_j^2 - \lambda_k^2)} \quad (4.1)$$

where $C = (2N - 1)! \prod_{j=1}^{2N-1} (2j - 1)!$, and W is the Weyl group, which consists here of permutations followed by reflections ($y_i \rightarrow \pm y_i$; $i = 1, \dots, N$) with an even number of sign changes.

In the appendix B, we give a more explicit determinantal expression for this Harish Chandra integral for the orthogonal group $\text{SO}(2N)$.

For the $p = 3$ critical model (3.5), let us compute the perturbation expansion of the free energy $F = \log Z$. From Th. 2, we obtain (see appendix B)

$$Z = \int \prod_{i=1}^k dy_i \frac{\prod (y_i^2 - y_j^2)}{\prod (\lambda_i^2 - \lambda_j^2)} e^{-\frac{1}{2} \sum y_i^4 - 2 \sum y_i \lambda_i} \quad (4.2)$$

To obtain the series in powers of $1/\lambda$, we change $\lambda_i \rightarrow \lambda_i^3$ and make a shift $y_i \rightarrow y_i - \lambda_i$ to eliminate the $y_i \lambda_i^3$ term. Then, the problem reduces to cubic and quartic perturbations with a Gaussian weight. For instance, $k = 2$ for $O(2k)$ case, we have from Th.2,

$$\begin{aligned} \int dy_1 dy_2 \frac{y_1^2 - y_2^2}{\lambda_1^2 - \lambda_2^2} e^{-\frac{1}{2}(y_1^4 + y_2^4) - 2(y_1 \lambda_1^3 + y_2 \lambda_2^3)} \\ = 1 + \frac{1}{72} \left(\frac{1}{\lambda_1^4} + \frac{1}{\lambda_2^4} \right) + \frac{1}{12} \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \right)^2 + O\left(\frac{1}{\lambda^8}\right) \end{aligned} \quad (4.3)$$

where we have dropped a normalization constant. In that simple $k = 2$ -case it is easy to make this calculation directly without use of Th.2, and the results do agree. The coefficients $\frac{1}{72}$ and $\frac{1}{12}$ are universal factors independent of k . For general k , it is useful to write the

Vandermonde product as a determinant with the y_i^2 , and the evaluation is straightforward. We introduce the following parameters, similar to those of the unitary case,

$$t_{n,j} = (p)^{\frac{j-p-n(p+2)}{2(p+1)}} \prod_{l=0}^{n-1} (lp + j + 1) \sum_{i=1}^k \frac{1}{\lambda_i^{pn+j+1}} \tag{4.4}$$

with here $p=3$. Note that the normalization of the $t_{n,j}$ is slightly different from the unitary case, in which p is replaced by $-p$ in the first factor of (4.4). However, this definition is not appropriate for the half-integer genus (non orientable surface), since the first factor becomes irrational number. So, we define in this half-integer genus case (condition will appear below in (4.9)) as

$$t_{n,j} = \prod_{l=0}^{n-1} (lp + j + 1) \sum_{i=1}^k \frac{1}{\lambda_i^{pn+j+1}}. \tag{4.5}$$

The index n_i stands for the power of the first Chern class c_1 . The index j is the spin index, which takes the values $j = 1, 2, \dots, p - 1$.

We end up with an expansion, similar to the unitary case, but with different coefficients

$$\log Z = \sum \langle \prod \tau_{n_i, j_i}^{d_{n_i, j_i}} \rangle \prod \frac{t_{n_i, j_i}^{d_{n_i, j_i}}}{d_{n_i, j_i}!} \tag{4.6}$$

For $p = 3$, the lowest orders of the $O(N)$ model are given by

$$\begin{aligned} \log Z = & \frac{1}{72} \sum \frac{1}{\lambda_i^4} + \frac{1}{12} \left(\sum \frac{1}{\lambda_i^2} \right)^2 + \frac{5}{432} \sum \frac{1}{\lambda_i^8} + \frac{1}{432} \left(\sum \frac{1}{\lambda_i^4} \right)^2 \\ & + \frac{1}{36} \left(\sum \frac{1}{\lambda_i^4} \right) \left(\sum \frac{1}{\lambda_i^2} \right)^2 - \frac{1}{108} \left(\sum \frac{1}{\lambda_i^2} \right)^4 + O\left(\frac{1}{\lambda^{12}}\right) \end{aligned} \tag{4.7}$$

Note that there is no odd-power of $1/\lambda$ such as $\sum \frac{1}{\lambda_i}$. This is due to the parity $\lambda_i \rightarrow -\lambda_i$ for real antisymmetric matrices of $O(2N)$. From the above series, the intersection numbers $\langle \prod_{n_i, j_i} \tau_{n_i, j_i}^{d_{n_i, j_i}} \rangle$ are obtained. In the unitary case, they are given by

$$\langle \tau_{n_1, j_1} \cdots \tau_{n_s, j_s} \rangle = \frac{1}{p^g} \int_{\bar{M}_{g,s}} \prod_{i=1}^s c_1(\mathcal{L}_i)^{n_i} C_T(j_1, \dots, j_s) \tag{4.8}$$

where c_1 is the first Chern class and $C_T(V)$ is the top Chern class [3]. In the present case, we call the intersection numbers as the coefficients of the expansion of $\log Z$ as (4.6).

The numbers of τ corresponds to the numbers of marked points s . The indices n_i, j_i are related to the genus g .

$$\sum_{i=1}^s \left(n_i + \frac{1}{p} j_i - 1 \right) = \left(3 - \left(1 - \frac{2}{p} \right) \right) (g - 1) \tag{4.9}$$

The genus g is given through an expansion in powers of the inverse of the size of the matrix, as is standard for matrix models. For this purpose, we introduce an overall factor k in the

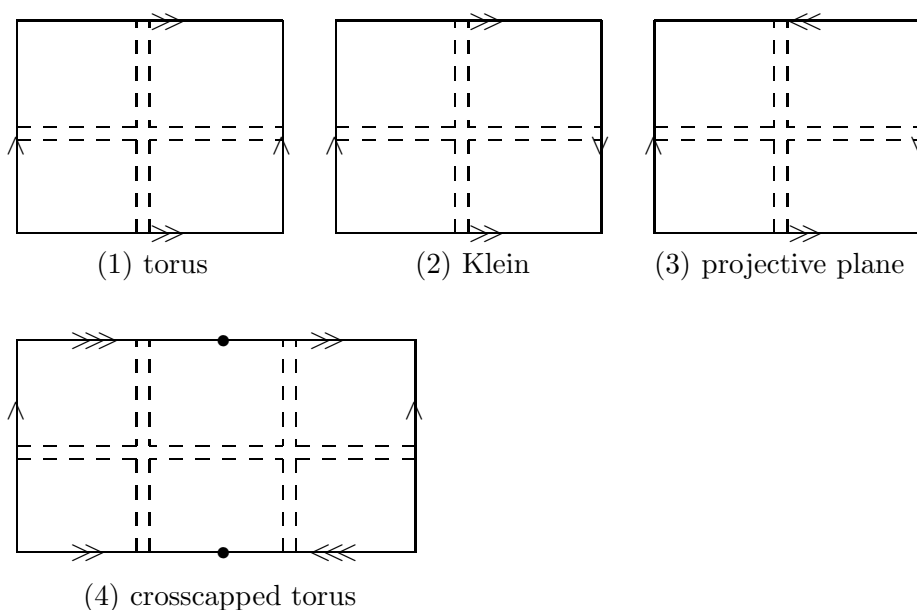


Figure 1: Opposite lines are glued together with matching arrow.

exponent (3.5), an integral over $k \times k$ matrices. Then $\log Z/k$ may be expanded in a series in powers of k^{2-2g} , with genus g . For the present antisymmetric matrix Y , odd powers of $1/k$ are also present. (In the unitary case, only even powers appear).

The genus g is given by the Euler characteristics,

$$V - E + F = 2 - 2(\text{type}) \tag{4.10}$$

where V , E and F are the numbers of vertices, edges and faces, respectively. For non-orientable surfaces, the genus g is replaced by the (type): we can still use g but it takes half-integer values. This definition coincides with that of (4.9). In figure 1, the lower order terms $\langle \tau_{1,0} \rangle_{g=1}$ ((1)torus and (2)Klein bottle), $\langle \tau_{0,1}^2 \rangle_{g=\frac{1}{2}}$ ((3)projective plane), and $\langle \tau_{2,1} \rangle_{g=\frac{3}{2}}$ (crosscapped torus) are depicted.

For non-orientable surface, new characteristic terms are present, such as $t_{1,0}$ for the Klein bottle, in addition to the torus, and $t_{0,1}^2$ (projective plane), $t_{2,1}$ (crosscapped torus) which did not exist in the unitary case. From (4.7), we have the intersection numbers,

$$\langle \tau_{1,0} \rangle_{g=1} = \frac{1}{24}, \quad \langle \tau_{0,1}^2 \rangle_{g=\frac{1}{2}} = \frac{1}{6}, \quad \langle \tau_{2,1} \rangle_{g=\frac{3}{2}} = \frac{1}{864}, \quad \langle \tau_{1,0}^2 \rangle_{g=0} = \frac{1}{24}. \tag{4.11}$$

5. Evolution operators at edge singularities

We have derived the higher Airy matrix model of (3.5) from the large N limit of the characteristic polynomials for the antisymmetric matrix Y , which is dual to the characteristic

polynomial of X . The choice of the $a_n = a_c = \sqrt{N}$ for (3.3) corresponds to a singular point in the spectrum of eigenvalues of X . At that critical point the density of states of X has a singularity at the origin. For $a > a_c$ there is a gap at the origin in the spectrum (whose support lies on the imaginary axis), and at $a = a_c$ this gap is closing. This happens also in the unitary case for an external matrix source with eigenvalues $\pm a$ [7, 8].

An integral representation for the evolution operators U for the vertices of $O(N)$ (N even) may be obtained from the Fourier transform of the correlation functions. The derivations are given in appendix C. The evolution operators $U(s_1, \dots, s_n)$ are defined as

$$U(s_1, \dots, s_n) = \frac{1}{N} \langle \text{tr} e^{s_1 X} \text{tr} e^{s_2 X} \dots \text{tr} e^{s_n X} \rangle_A \quad (5.1)$$

For the one point function of $(2N) \times (2N)$ antisymmetric matrix X , we have from (C.11),

$$U(s) = \frac{1}{2N} \langle \text{tr} e^{sX} \rangle_A = -\frac{1}{Ns} \oint \frac{dv}{2\pi i} \prod_{i=1}^N \frac{v^2 + a_i^2}{(v + \frac{s}{2})^2 + a_i^2} \left(\frac{v + \frac{s}{2}}{v + \frac{s}{4}} \right) e^{sv + \frac{s^2}{4}} \quad (5.2)$$

For a $N \times N$ real antisymmetric matrix X , the sourceless probability density $A = 0$,

$$P(X) = \frac{1}{Z} e^{\gamma \text{tr} X^2}, \quad Z = \left(\frac{\pi}{2\gamma} \right)^{\frac{N(N-1)}{4}} \quad (5.3)$$

gives the expectation values

$$\begin{aligned} \langle X_{ij} X_{kl} \rangle &= -\frac{1}{4\gamma} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \\ \langle \text{tr} X^2 \rangle &= -\frac{N(N-1)}{4\gamma} \\ \langle (\text{tr} X^2)^2 \rangle &= \frac{N(N-1)(N^2 - N + 4)}{16\gamma^2} \\ \langle \text{tr} X^4 \rangle &= \frac{N(N-1)(2N-1)}{16\gamma^2} \end{aligned} \quad (5.4)$$

We have thus to compare

$$U(s) = \frac{1}{N} \langle \text{tr} e^{sX} \rangle = 1 + \frac{s^2}{2N} \langle \text{tr} X^2 \rangle + \frac{s^4}{4!N} \langle \text{tr} X^4 \rangle + \dots \quad (5.5)$$

with this integral representation. For instance, in the case of $N=1$, (and $\gamma = 1/2$) the formula (5.2), after taking the residue at $v = -\frac{s}{2}$, leads to

$$U(s) = e^{-\frac{s^2}{4}} = 1 - \frac{s^2}{4} + \frac{s^4}{32} + \dots \quad (5.6)$$

which indeed agrees with $N = 2$ in (5.4),

$$\begin{aligned} \frac{1}{N} \langle \text{tr} X^2 \rangle &= -\frac{(N-1)}{2} \Big|_{N=2} = -\frac{1}{2}, \\ \frac{1}{N} \langle \text{tr} X^4 \rangle &= \frac{(N-1)(2N-1)}{4} \Big|_{N=2} = \frac{3}{4} \end{aligned} \quad (5.7)$$

For $N = 2, 3, \dots$, it is easily verified that the integral representation of $U(s)$ agrees with (5.4).

In a previous article, we have found an explicit formula giving the zero-replica limit $N \rightarrow 0$ for $U(s_1, \dots, s_n)$ in the unitary case, in the absence of an external source ($A = 0$) [5]. There it was shown that

$$\lim_{N \rightarrow 0} U(s_1, \dots, s_n) = \frac{1}{\sigma^2} \prod_{i=1}^n 2 \operatorname{sh} \frac{s_i \sigma}{2} \tag{5.8}$$

where $\sigma = \sum_{i=1}^n s_i$. For $n=1$, this is simply

$$\lim_{N \rightarrow 0} U(s) = \frac{2}{s^2} \operatorname{sh} \frac{s^2}{2} = 1 + \frac{s^4}{24} + \frac{s^8}{5! \cdot 2^4} + \dots \tag{5.9}$$

This means that

$$\lim_{N \rightarrow 0} \frac{1}{N} \langle \operatorname{tr} M^4 \rangle = 1. \tag{5.10}$$

The replica limit counts the numbers of diagrams which can be drawn in one stroke line, and it corresponds to one marked point for the intersection numbers.

For real antisymmetric matrices X , this replica limit of $U(s)$ is different from the previous unitary case. From (5.2), we obtain

$$\begin{aligned} \lim_{N \rightarrow 0} U(s) &= \frac{2}{s} \oint \frac{dv}{2\pi i} \log \left(1 + \frac{s}{2v} \right) \left(\frac{v + \frac{s}{2}}{v + \frac{s}{4}} \right) e^{sv + \frac{s^2}{4}} \\ &= \frac{4}{s^2} \operatorname{sh} \frac{s^2}{4} + \int_0^{\frac{s^2}{4}} dx \frac{\operatorname{sh} x}{x} \\ &= 1 + \frac{s^2}{4} + \frac{s^4}{96} + \frac{s^6}{1152} + \dots \end{aligned} \tag{5.11}$$

The coefficients of s^2 and s^4 , $\frac{1}{4}$ and $\frac{1}{96}$, are consistent with (5.5) by (5.4), when we take $\gamma = \frac{1}{2}$ and $N \rightarrow 0$. The term of order s^2 is a Möbius band (projective plane), and it is a typical non-orientable surface. This term comes from the integral of (5.11), which does not exist in the unitary case.

We obtain the connected part of the two-point correlation, after the shift $u_i \rightarrow u_i + \frac{s_i}{2}$ in (C.19),

$$\begin{aligned} \lim_{N \rightarrow 0} \tilde{U}(s_1, s_2) &= -e^{\frac{1}{4}(s_1^2 + s_2^2)} \oint \frac{du_1 du_2}{2\pi i} e^{s_1 u_1 + s_2 u_2} \\ &\quad \times \log \left(1 + \frac{s_1}{2u_1} \right) \frac{(u_1 + \frac{s_1}{2})(u_2 + \frac{s_2}{2})}{[u_1^2 - (u_2 + \frac{s_2}{2})^2][u_2^2 - (u_1 + \frac{s_1}{2})^2]} \end{aligned} \tag{5.12}$$

with

$$U(s_1, s_2) = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \tilde{U}(\epsilon_1 s_1, \epsilon_2 s_2). \tag{5.13}$$

We deform the contour of u_2 near the origin to the poles (i) $u_2 = u_1 + \frac{s_1}{2}$, (ii) $u_2 = -u_1 - \frac{s_1}{2}$, (iii) $u_2 = -u_1 - \frac{s_2}{2}$, (iv) $u_2 = u_1 - \frac{s_2}{2}$. Then we obtain in the replica limit $N \rightarrow 0$,

$$\begin{aligned} \lim_{N \rightarrow 0} \tilde{U}(s_1, s_2) &= \frac{2}{(s_1 + s_2)^2} \text{sh} \frac{s_1}{4} (s_1 + s_2) \text{sh} \frac{s_2}{4} (s_1 + s_2) \\ &\quad - \frac{2}{(s_1 - s_2)^2} \text{sh} \frac{s_1}{4} (s_1 - s_2) \text{sh} \frac{s_2}{4} (s_1 - s_2) \\ &\quad + \frac{1}{2} \int_{s_1 - s_2}^{s_1 + s_2} dy \frac{1}{y} \text{sh} \frac{s_1 y}{4} \text{sh} \frac{s_2 y}{4}. \end{aligned} \tag{5.14}$$

Since this is invariant under the change of signs of s_i , we have from (5.13),

$$\lim_{N \rightarrow 0} U(s_1, s_2) = 4 \lim_{N \rightarrow 0} \tilde{U}(s_1, s_2). \tag{5.15}$$

For $n > 2$ we operate in a similar fashion and obtain the following result which generalizes the theorem for the unitary case [5]:

Theorem 3.

$$\begin{aligned} \lim_{N \rightarrow 0} U(s_1, \dots, s_n) &= \sum_{\epsilon_i = \pm 1} W(\epsilon_1 s_1, \epsilon_2 s_2, \dots, \epsilon_n s_n), \\ W(s_1, \dots, s_n) &= \frac{1}{2\sigma^2} \prod_{i=1}^n \left(4 \text{sh} \frac{s_i \sigma}{4} \right) + \frac{1}{2} \int_0^\sigma dy \frac{1}{y} \prod_{i=1}^n \text{sh} \frac{s_i y}{4} \end{aligned} \tag{5.16}$$

where $\sigma = s_1 + \dots + s_n$.

6. Intersection numbers from $U(s_1, \dots, s_n)$

For one marked point, we consider the evolution operator $U(s)$ in an external source A , chosen at a critical value. We discuss here the case $p=3$. From (5.2), by the scalings of $v \rightarrow \sqrt{N}v, s \rightarrow s/\sqrt{N}$, and by the critical value $a_i^2 = N$, we have

$$U(s) = -\frac{1}{s} e^{\frac{s^2}{4N}} \oint \frac{dv}{2\pi i} \left(\frac{1 + v^2}{1 + (v + \frac{s}{2N})^2} \right)^N \frac{v + \frac{s}{2N}}{v + \frac{s}{4N}} e^{sv} \tag{6.1}$$

Exponentiating the term of power N , we have

$$\begin{aligned} -N \log \left[1 + \left(v + \frac{s}{2N} \right)^2 \right] + N \log(1 + v^2) + sv + \frac{s^2}{4N} \\ = v^3 s + \frac{3}{4N} v^2 s^2 + \frac{1}{4N^2} v s^3 + \frac{1}{32N^3} s^4 + \dots \end{aligned} \tag{6.2}$$

The first four terms are of order N after rescaling $s \rightarrow Ns$. Further, by the replacement by $s \rightarrow \sqrt{2}s, N \rightarrow 2N, v \rightarrow u/\sqrt{2}$, we obtain in the large N limit,

$$U(s) = -\frac{e^{\frac{N}{4}s^4}}{Ns} \int \frac{du}{2\pi i} e^{Nsu^3 + \frac{3}{2}Ns^2u^2 + Ns^3u} \left(\frac{u + s}{u + \frac{s}{2}} \right) \tag{6.3}$$

The shift $u \rightarrow u - \frac{s}{2}$, and $u = \frac{1}{s^{1/3}}t$ gives

$$U(s) = -\frac{1}{Ns^{4/3}} \int \frac{dt}{2\pi i} e^{Nt^3 + \frac{N}{4}s^{8/3}t} \left(1 + \frac{s^{4/3}}{2t}\right). \quad (6.4)$$

We further make a scale $t \rightarrow -it/(3N)^{1/3}$, then we have

$$U(s) = \frac{1}{3^{1/3}(Ns)^{4/3}} \left[\frac{1}{\pi} \int_0^\infty dt \cos\left(\frac{t^3}{3} + xt\right) - \frac{1}{\pi} \frac{s^{4/3}(3N)^{1/3}}{2} \int_0^\infty dt \frac{1}{t} \sin\left(\frac{t^3}{3} + xt\right) \right]. \quad (6.5)$$

where $x = -N^{2/3}s^{8/3}/(4 \cdot 3^{1/3})$. Using the Airy function $A_i(x)$,

$$A_i(x) = \frac{1}{\pi} \int_0^\infty dt \cos\left(\frac{t^3}{3} + xt\right) \quad (6.6)$$

we obtain

$$U(s) = \frac{1}{3^{1/3}(Ns)^{4/3}} \left(A_i(x) - \frac{s^{4/3}(3N)^{1/3}}{2} \int_0^x dx' A_i(x') \right) \quad (6.7)$$

Since s is a Fourier transform variable, it is proportional to

$$s \sim \frac{1}{\lambda^3}, \quad x \sim s^{8/3} \sim \frac{1}{\lambda^8} \quad (6.8)$$

The Airy function $A_i(x)$ has asymptotic expansion

$$A_i(x) = A_i(0) \left(1 + \frac{1}{3!}x^3 + \frac{1 \cdot 4}{6!}x^6 + \dots + \frac{1 \cdot 4 \cdot 7}{9!}x^9 + \dots \right) + A_i'(0) \left(x + \frac{2}{4!}x^4 + \frac{2 \cdot 5}{7!}x^7 + \frac{2 \cdot 5 \cdot 8}{10!}x^{10} + \dots \right) \quad (6.9)$$

where $A_i(0) = 3^{-2/3}/\Gamma(2/3)$ and $A_i'(0) = -3^{-1/3}/\Gamma(1/3)$.

Therefore, we obtain the intersection numbers from the coefficients of the evolution operator $U(s)$ for the one marked point like in the previous unitary case [5, 6]. The first series of expansion of $A_i(x)$ in (6.9) gives the intersection numbers for the spin $j = 1$, and the second series gives the intersection numbers for the spin $j = 0$. From the first term in (6.7) with (6.9), one finds the intersection numbers $\langle \tau_{n,j} \rangle_g$ for integer g

$$\langle \tau_{(8g-5-j)/3,j} \rangle_g = \frac{1}{(24)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})} \quad (6.10)$$

where $j = 0$ for $g = 3l + 1$ and $j = 1$ for $g = 3l$. The intersection numbers for non orientable surfaces of half-integer genera, are obtained from the second term of (6.7), namely the integral of the Airy function. Taking into account the normalization (4.5), this leads to explicit results, such as

$$\langle \tau_{2,1} \rangle_{g=3/2} = \frac{1}{864}$$

7. Conclusion

We have derived the intersection numbers for non-orientable surfaces from generalized Kontsevich Airy integrals over random antisymmetric matrices, the Lie algebra of the group $SO(2N)$. An $N - k$ duality between k -point functions in $N \times N$ Gaussian matrix integrals, and N -point functions for $k \times k$ integrals, in the presence of an external matrix source, allows one to relate those generalized Airy integrals to the edge behavior of Gaussian models. Those Gaussian models are then much easier to deal with than the original integrals. The existence for Lie algebras of classical groups (such as Hermitian matrices for $U(N)$ or antisymmetric matrices for $O(N)$) of an Harish Chandra integral over the group elements is a key ingredient in these calculations. These techniques should be useful for characterizing the geometric properties of those non orientable surfaces, and for comparing zeros of analytic functions to random matrix spectra.

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A. Derivation of theorem 1

Let us begin with the simple case, $k=1$. The determinant is given by the integral,

$$\langle \det(\lambda \cdot I - X) \rangle = \left\langle \int dc_a d\bar{c}_a e^{\bar{c}_a(\lambda \cdot I - X)_{ab}c_b} \right\rangle \quad (\text{A.1})$$

over the $2N$ Grassmann variables c_a, \bar{c}_a . The probability measure $P(X)$ for the average is

$$P(X) = \frac{1}{Z_A} e^{\frac{1}{2}\text{tr}X^2 + \text{tr}XA} \quad (\text{A.2})$$

Absorbing the antisymmetric part of the term $\bar{c}_a X_{ab} c_b$ in the external source, $A \rightarrow A'$, with

$$A'_{ab} = A_{ab} - \frac{1}{2}(\bar{c}_a c_b - \bar{c}_b c_a) \quad (\text{A.3})$$

We obtain

$$\text{tr}A'^2 = \text{tr}A^2 - 2A_{ab}\bar{c}_b c_a - \frac{1}{2}(\bar{c}_a c_a)(\bar{c}_b c_b) \quad (\text{A.4})$$

where a and b run over $1, 2, \dots, N$. Writing the term $(\bar{c}_a c_a)(\bar{c}_b c_b)$ in the exponent as

$$e^{-\frac{1}{4}(\bar{c}_a c_a)(\bar{c}_b c_b)} = \frac{1}{\sqrt{\pi}} \int dy e^{-y^2 + iy\bar{c}c} \quad (\text{A.5})$$

we obtain

$$\begin{aligned} \langle \det(\lambda \cdot I - X) \rangle &= \frac{1}{\sqrt{\pi}} \int dy \int dcd\bar{c} e^{-y^2 + (\lambda + iy)\bar{c}_a c_a - A_{ab}\bar{c}_b c_a} \\ &= \frac{1}{\sqrt{\pi}} \int dy e^{-y^2} \prod_{j=1}^N \det \left((\lambda + iy) \cdot 1 - \frac{a_j}{\gamma} i\sigma_2 \right) \\ &= \frac{1}{\sqrt{\pi}} \int dy e^{-y^2} \prod_{j=1}^N [(\lambda + iy)^2 - a_j^2] \end{aligned} \quad (\text{A.6})$$

where σ_2 is the second Pauli matrix. On the other hand for the one-point function the right-hand side in (2.1) (theorem 1) is an integral over one single real number

$$\begin{aligned} \left\langle \prod_{j=1}^N \det(a_j \cdot \mathbf{I} - Y) \right\rangle_{\Lambda} &= \frac{1}{Z_{\Lambda}} \int \prod_{j=1}^N \det \begin{pmatrix} a_j & y \\ -y & a_j \end{pmatrix} e^{-y^2 - 2i\lambda y} \\ &= \frac{1}{\sqrt{\pi}} e^{-\lambda^2} \int dy \prod_{j=1}^N (a_j^2 + y^2) e^{-y^2 - 2i\lambda y}. \end{aligned} \quad (\text{A.7})$$

The shift $iy \rightarrow iy + \lambda$ shows that (A.6) and (A.7) are identical.

When $k \geq 2$ for averaging the k characteristic polynomials, $k \times 2N$ Grassmann variables \bar{c}_a^{α} and c_b^{α} are necessary ($\alpha = 1, \dots, k$; $a, b = 1 \cdots N$). As for $k = 1$, we have

$$A'_{ab} = A_{ab} - \frac{1}{2}(\bar{c}_a^{\alpha} c_b^{\alpha} - \bar{c}_b^{\alpha} c_a^{\alpha}) \quad (\text{A.8})$$

$$\text{tr} A'^2 = -2A_{ba} \bar{c}_a^{\alpha} c_b^{\alpha} - \frac{1}{2} \bar{c}_a^{\alpha} c_a^{\beta} \bar{c}_b^{\beta} c_b^{\alpha} - \frac{1}{2} \bar{c}_a^{\alpha} \bar{c}_a^{\beta} c_b^{\beta} c_b^{\alpha} \quad (\text{A.9})$$

The last two terms are replaced by the following integrals,

$$e^{-\frac{1}{4\gamma}(\bar{c}_a^{\alpha} c_a^{\beta})(\bar{c}_b^{\beta} c_b^{\alpha})} = \int dB e^{-\gamma \text{tr} B^2 + i \text{tr} B^{\beta\alpha} \bar{c}_a^{\alpha} c_a^{\beta}} \quad (\text{A.10})$$

$$e^{-\frac{1}{4\gamma}(\bar{c}_a^{\alpha} \bar{c}_a^{\beta})(c_b^{\beta} c_b^{\alpha})} = \int dD dD^* e^{\gamma \text{tr} D^* D + \frac{1}{2} \bar{c}_a^{\alpha} \bar{c}_a^{\beta} D^{\beta\alpha} + \frac{1}{2} (D^*)^{\alpha\beta} c_b^{\beta} c_b^{\alpha}} \quad (\text{A.11})$$

where B is a $k \times k$ Hermitian matrix and D is a $k \times k$ antisymmetric complex matrix. Thus we obtain,

$$\begin{aligned} \left\langle \prod_{\alpha=1}^k \det(\lambda_{\alpha} - X) \right\rangle &= \int dB dD e^{-\gamma \text{tr} B^2 + \gamma \text{tr} D^* D - \frac{1}{\gamma} A_{ba} \bar{c}_b^{\alpha} c_a^{\alpha}} \\ &\quad \times e^{\lambda_{\alpha} \bar{c}_a^{\alpha} c_a^{\alpha} + i B^{\alpha\beta} \bar{c}_a^{\beta} c_a^{\alpha} + \frac{1}{2} D_{\alpha\beta} \bar{c}_a^{\beta} \bar{c}_a^{\alpha} + \frac{1}{2} D_{\alpha\beta}^* c_b^{\beta} c_b^{\alpha}} \end{aligned} \quad (\text{A.12})$$

The exponent of this integral is a quadratic form in the Grassmann variables.

Let us first consider the $k = 2$ case. The exponent is of the form $\sum_n \Psi_n^t M_n \Psi_n$, where

$$\Psi_n^t = (\bar{c}_{2n+1}^1, \bar{c}_{2n+2}^1, \bar{c}_{2n+1}^2, \bar{c}_{2n+2}^2, c_{2n+1}^1, c_{2n+2}^1, c_{2n+1}^2, c_{2n+2}^2) \quad (\text{A.13})$$

with the 8 by 8 matrix M_n

$$M_n = \begin{pmatrix} \tilde{D} & \tilde{B}_n \\ -\tilde{B}_n^t & \tilde{D}^* \end{pmatrix} \quad (\text{A.14})$$

where

$$\tilde{D} = \begin{pmatrix} 0 & 0 & D_{21} & 0 \\ 0 & 0 & 0 & D_{21} \\ -D_{21} & 0 & 0 & 0 \\ 0 & -D_{21} & 0 & 0 \end{pmatrix} \quad (\text{A.15})$$

$$\tilde{B}_n = \begin{pmatrix} \lambda_1 + iB_{11} & \frac{i}{\gamma} a_n & iB_{21} & 0 \\ -\frac{i}{\gamma} a_n & \lambda_1 + iB_{11} & 0 & iB_{21} \\ iB_{12} & 0 & \lambda_2 + iB_{22} & \frac{1}{\gamma} a_n \\ 0 & iB_{12} & -\frac{i}{\gamma} a_n & \lambda_2 + iB_{22} \end{pmatrix} \quad (\text{A.16})$$

Since the matrix M_n is antisymmetric, the Gaussian integral over the Ψ_n is the Pfaffian:

$$\begin{aligned} \text{Pf}(M_n) &= \left[|D_{21}|^2 - (\lambda_1 + iB_{11})(\lambda_2 + iB_{22}) - |B_{12}|^2 \right]^2 \\ &\quad + \frac{1}{\gamma^2} a_n^2 \left[-(\lambda_1 + iB_{11})^2 - (\lambda_2 + iB_{22})^2 + 2|B_{12}|^2 + 2|D_{21}|^2 \right] \\ &\quad + \frac{1}{\gamma^4} a_n^4 \end{aligned} \tag{A.17}$$

Writing $B_{11} = b_1, B_{22} = b_2, B_{12} = b_3 + ib_4, D = d_1 + id_2$, we define the real antisymmetric matrix Y as

$$Y = \begin{pmatrix} 0 & b_1 & b_4 + d_2 & b_3 + d_1 \\ -b_1 & 0 & d_1 - b_3 & b_4 - d_2 \\ -b_4 - d_2 & -d_1 + b_3 & 0 & b_2 \\ -b_3 - d_1 & -b_4 + d_2 & -b_2 & 0 \end{pmatrix} \tag{A.18}$$

which satisfies the following equation,

$$\det \left(\frac{a_n}{\gamma} \cdot \mathbf{I} - Y \right) = \text{Pf}(M_n) \tag{A.19}$$

Since

$$\begin{aligned} \text{tr} Y^2 &= -2(b_1^2 + b_2^2) - 4(b_3^2 + b_4^2 + d_1^2 + d_2^2) \\ &= -\text{tr} B^2 - \text{tr} D^* D \end{aligned} \tag{A.20}$$

theorem 1 holds.

For $k > 2$, the same procedure leads to the expression of an antisymmetric matrix Y , which is made of block of 2×2 matrices, given by

$$Y_{ij} = (\text{Im} B_{ij}) \cdot \mathbf{I} + (i\sigma_1) \text{Re} B_{ij} + \sigma_1 \text{Im} D_{ij} + \sigma_3 \text{Re} D_{ij} \tag{A.21}$$

where the σ_i are the 2×2 Pauli matrices.

B. Harish Chandra integral formula of theorem 2

The Weyl group W of the $\text{SO}(2N)$ Lie algebra is the permutation group S_{2N} followed by reflection symmetries $y_i \rightarrow \epsilon_i y_i$ ($\epsilon_i = \pm 1$), with an even number of $\epsilon_i = -1$. Then the sum over the elements of the Weyl group contained in the numerator of (4.1) becomes

$$I = \sum_{\substack{\epsilon_1 = \pm 1, \dots, \epsilon_N = \pm 1 \\ \epsilon_1 \epsilon_2 \dots \epsilon_N = 1}} \sum_{\sigma \in S_N} (\det \sigma) \exp \left[2 \sum_{j=1}^N \epsilon_{\sigma(j)} y_{\sigma(j)} \lambda_j \right] \tag{B.1}$$

where the sum is restricted to reflections with an even number of sign changes, i.e.

$$\epsilon_1 \epsilon_2 \dots \epsilon_N = 1. \tag{B.2}$$

Since $\det\sigma = (-1)^{|\sigma|}$ (in which $|\sigma|$ is the parity of the permutation), the sum over the $N!$ elements σ of S_N is a determinant. Therefore we obtain

$$I = \sum_{\substack{\epsilon_1=\pm 1, \dots, \epsilon_N=\pm 1 \\ \epsilon_1 \epsilon_2 \dots \epsilon_N = 1}} \det \left[e^{2\epsilon_i y_i \lambda_j} \right] \quad (\text{B.3})$$

The result in this form is sufficient for the purposes of section 4. But one may go a bit further. Writing for each matrix element $e^{2\epsilon_i y_i \lambda_j} = \cosh(2y_i \lambda_j) + \epsilon_i \sinh(2y_i \lambda_j)$ we obtain a sum of 2^N determinants weighted by products of ϵ_i . The sum over those ϵ_i , restricted by the condition (B.2), leads to a cancellation of all the terms except two. The final result is

$$I = 2^{N-1} (\det[\cosh(2y_i \lambda_j)] + \det[\sinh(2y_i \lambda_j)]) \quad (\text{B.4})$$

For instance, for $N = 2$ the signs are $(\epsilon_1, \epsilon_2) = (1, 1)$ or $(-1, -1)$. The sum over these terms gives

$$\begin{aligned} I &= \det[e^{2y_i \lambda_j}] + \det[e^{-2y_i \lambda_j}] \\ &= (e^{2y_1 \lambda_1 + 2y_2 \lambda_2} - e^{2y_1 \lambda_2 + 2y_2 \lambda_1}) + (e^{-2y_1 \lambda_1 - 2y_2 \lambda_2} - e^{-2y_1 \lambda_2 - 2y_2 \lambda_1}). \end{aligned} \quad (\text{B.5})$$

which is indeed identical to

$$I = 2(\det[\cosh(2y_i \lambda_j)] + \det[\sinh(2y_i \lambda_j)]). \quad (\text{B.6})$$

Let us apply the result (B.3) to the integral (3.5):

$$Z = \int dY e^{-\frac{1}{p+1} \text{tr} Y^{p+1} + \text{tr} Y \Lambda} \quad (\text{B.7})$$

in which Y runs over the $2N \times 2N$ antisymmetric matrices, the Lie algebra of $\text{SO}(2N)$. One may use the rotational invariance of the measure to write $Y = g y g^{-1}$ in which g is an element of $\text{SO}(2N)$ and y is a canonical matrix (2.2), namely

$$y = y_1 v \oplus \dots \oplus y_N v, \quad v = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The integral over Y may be replaced by an integral over g and over the y_i 's. The Jacobian is, up to a constant factor, $J = \prod_{i < j} (y_i^2 - y_j^2)^2$. Using the Harish Chandra integral, one integrates over g and, using the result (B.3), one obtains

$$Z = \sum_{\substack{\epsilon_1=\pm 1, \dots, \epsilon_N=\pm 1 \\ \epsilon_1 \epsilon_2 \dots \epsilon_N = 1}} \int dy_1 \dots dy_N \frac{\prod (y_i^2 - y_j^2)}{\prod (\lambda_i^2 - \lambda_j^2)} e^{-\frac{2}{p+1} \sum_1^N y_i^{p+1}} \det[e^{2\epsilon_i y_i \lambda_j}]. \quad (\text{B.8})$$

Since p is odd, one can change $\epsilon_i y_i \rightarrow y_i$. The antisymmetry of $\prod (y_i^2 - y_j^2)$ under permutations allows one to replace the determinant by its diagonal term $\exp(2 \sum y_i \lambda_i)$ (up to a factor $N!$). The restricted sum over the ϵ_i is simply the factor 2^{N-1} . We are thus led to the integral (4.2) of section 4.

C. Integral representations for $U(s_1, \dots, s_n)$

For real antisymmetric matrices, the support of the density of states $\rho(\lambda)$ is the imaginary axis,

$$\rho(\lambda) = \frac{1}{N} \langle \text{tr} \delta(\lambda - X) \rangle. \quad (\text{C.1})$$

and it is an even function of λ . The Fourier transform of $\rho(\lambda)$ is $U(t)$,

$$\begin{aligned} U(t) &= \int d\lambda e^{it\lambda} \rho(\lambda) \\ &= \frac{1}{N} \langle \text{tr} e^{itX} \rangle. \end{aligned} \quad (\text{C.2})$$

Through an orthogonal transformation g ($g \in \text{SO}(2N)$), one can bring the matrix X to the canonical form

$$X = x_1 v \oplus \dots \oplus x_N v, \quad v = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{C.3})$$

from which one has

$$\text{tr} e^{sX} = 2 \sum_{i=1}^N \cos(sx_i) \quad (\text{C.4})$$

From theorem 2, the evolution operator $U(s)$ becomes

$$U(s) = \frac{1}{N} \sum_{\alpha=1}^N \int \prod_i dx_i \cos(sx_\alpha) \frac{\Delta(x_j^2)}{\Delta(a_j^2)} e^{-\sum x_i^2 + 2 \sum a_j x_j} \quad (\text{C.5})$$

If we write $\cos(sx_\alpha) = \text{Re}(e^{isx_\alpha})$ the integral amounts to an integral over antisymmetric matrices with the shifted source $a_j \rightarrow \tilde{a}_j = a_j + \frac{1}{2}is\delta_{j,\alpha}$. Then the integrals over the x_i 's are just the normalization for Gaussian antisymmetric matrices in the source \tilde{A} :

$$\int dX e^{\frac{1}{2}\text{Tr}X^2 + \text{Tr}\tilde{A}X} = e^{-\frac{1}{2}\text{Tr}\tilde{A}^2}. \quad (\text{C.6})$$

Indeed using again the Harish Chandra theorem, the left-hand side is simply

$$\int dX e^{\frac{1}{2}\text{Tr}X^2 + \text{Tr}\tilde{A}X} = \int \prod_i dx_i \frac{\Delta(x_j^2)}{\Delta(\tilde{a}_j^2)} e^{-\sum x_i^2 + 2 \sum \tilde{a}_j x_j} \quad (\text{C.7})$$

which provides the result that we need:

$$\int \prod_i dx_i \Delta(x_j^2) e^{-\sum x_i^2 + 2 \sum a_j x_j + isx_\alpha} = \frac{\Delta(\tilde{a}_j^2)}{\Delta(a_j^2)} e^{-\frac{1}{2}\text{Tr}\tilde{A}^2 + \frac{1}{2}\text{Tr}A^2}. \quad (\text{C.8})$$

This leads to

$$U(s) = \frac{1}{2N} \sum_{\alpha=1}^N \prod_{\gamma \neq \alpha}^N \left(\frac{(a_\alpha + \frac{is}{2})^2 - a_\gamma^2}{a_\alpha^2 - a_\gamma^2} \right) e^{isa_\alpha - \frac{s^2}{4}} + (s \rightarrow -s). \quad (\text{C.9})$$

The normalization is such that $U(0) = 1$. It is useful to express (C.9) as a contour integral,

$$U(s) = \frac{1}{Ns} \oint \frac{du}{2\pi i} \prod_{\gamma=1}^N \left(\frac{(u + \frac{is}{2})^2 - a_\gamma^2}{u^2 - a_\gamma^2} \right) \frac{u}{iu - \frac{s}{4}} e^{ius - \frac{s^2}{4}} \quad (\text{C.10})$$

where the contour encircles the poles $u = a_\gamma$.

Changing variables, $u + \frac{is}{2} = -iv$, one obtains $U(s)$

$$U(s) = -\frac{1}{Ns} \oint \frac{dv}{2\pi i} \prod_{n=1}^N \left(\frac{v^2 + a_n^2}{(v + \frac{s}{2})^2 + a_n^2} \right) \frac{v + \frac{s}{2}}{v + \frac{s}{4}} e^{vs + \frac{s^2}{4}}, \quad (\text{C.11})$$

which is the representation that we have used in (5.2).

The two-point correlation function $U(s_1, s_2)$ is given by Th.2,

$$\begin{aligned} U(s_1, s_2) &= \frac{1}{2N} \langle \text{tr} e^{s_1 X} \text{tr} e^{s_2 X} \rangle \\ &= \frac{2}{N} \sum_{\alpha_1, \alpha_2=1}^N \int \prod_{i=1}^N dx_i \cos(s_1 x_{\alpha_1}) \cos(s_2 x_{\alpha_2}) \frac{\Delta(x^2)}{\Delta(a^2)} e^{-\sum x_i^2 + 2\sum a_j x_j}. \end{aligned} \quad (\text{C.12})$$

We make replacements of $\cos(is_i x_{\alpha_i})$ by $\frac{1}{2} e^{is_i x_{\alpha_i}}$, ($i = 1, 2$) in (C.12), and we name it as $\tilde{U}(s_1, s_2)$. Then, we have

$$U(s_1, s_2) = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \tilde{U}(\epsilon_1 s_1, \epsilon_2 s_2). \quad (\text{C.13})$$

The sum in (C.12) is divided into two parts, $\sum_{\alpha_1 = \alpha_2}$ and $\sum_{\alpha_1 \neq \alpha_2}$. The first part can be neglected. By the double contour integrals, the double sum is expressed as

$$\begin{aligned} \tilde{U}(s_1, s_2) &= \frac{1}{2Ns_1 s_2} \oint \frac{dudv}{(2\pi i)^2} \prod_{\gamma=1}^N \left(\frac{(u + \frac{is_1}{2})^2 - a_\gamma^2}{u^2 - a_\gamma^2} \right) \left(\frac{(v + \frac{is_2}{2})^2 - a_\gamma^2}{v^2 - a_\gamma^2} \right) \\ &\times \frac{uv}{(iu - \frac{s_1}{4})(iv - \frac{s_2}{4})} \frac{[(u + \frac{is_1}{2})^2 - (v + \frac{is_2}{2})^2](u^2 - v^2)}{[(u + \frac{is_1}{2})^2 - v^2][u^2 - (v + \frac{is_2}{2})^2]} e^{ius_1 + ivs_2 - \frac{1}{4}(s_1^2 + s_2^2)} \end{aligned} \quad (\text{C.14})$$

By the Cauchy determinant identity,

$$\det \frac{1}{x_i^2 - y_j^2} = (-1)^{n(n-1)/2} \frac{\prod_{i < j} (x_i^2 - x_j^2)(y_i^2 - y_j^2)}{\prod_{i,j} (x_i^2 - y_j^2)} \quad (\text{C.15})$$

with $x_i = u_i + \frac{is_i}{2}$, $y_i = u_i$, above expression is simplified as

$$\begin{aligned} \tilde{U}(s_1, s_2) &= \frac{1}{2N} e^{-\frac{1}{4}(s_1^2 + s_2^2)} \oint \frac{du_i}{(2\pi i)^2} e^{\sum is_i u_i} \prod_{i=1}^2 u_i \\ &\times \prod_{\gamma=1}^N \prod_{i=1}^2 \left(\frac{(u_i + \frac{is_i}{2})^2 - a_\gamma^2}{u_i^2 - a_\gamma^2} \right) \det \frac{1}{(u_i + \frac{is_i}{2})^2 - u_j^2} \end{aligned} \quad (\text{C.16})$$

For general n , we have similarly,

$$\begin{aligned} \tilde{U}(s_1, \dots, s_n) &= \frac{1}{2N} e^{-\frac{1}{4} \sum s_i^2} \oint \prod_{i=1}^N \frac{du_i}{2\pi i} e^{\sum i s_i u_i} \prod_{i=1}^n u_i \\ &\times \prod_{\gamma=1}^N \prod_{i=1}^n \left(\frac{(u_i + \frac{i s_i}{2})^2 - a_\gamma^2}{u_i^2 - a_\gamma^2} \right) \det \frac{1}{(u_i + \frac{i s_i}{2})^2 - u_j^2}. \end{aligned} \quad (\text{C.17})$$

with

$$U(s_1, \dots, s_n) = \sum_{\epsilon_i = \pm 1} \tilde{U}(\epsilon_1 s_1, \dots, \epsilon_n s_n). \quad (\text{C.18})$$

This expression reduces to the previous one in (C.10) for $n=1$. We need the connected part of $U(s_1, \dots, s_n)$, which is easily obtained from the expression of the determinant in (C.17). For $n=2$, we obtain by the change $u_i \rightarrow -i u_i$,

$$\begin{aligned} \tilde{U}_c(s_1, s_2) &= \frac{1}{2N} e^{-\frac{1}{4}(s_1^2 + s_2^2)} \oint \frac{du_i}{(2\pi i)^2} e^{\sum s_i u_i} \\ &\times \prod_{\gamma=1}^N \prod_{i=1}^2 \left(\frac{(u_i - \frac{s_i}{2})^2 + a_\gamma^2}{u_i^2 + a_\gamma^2} \right) \frac{u_1 u_2}{[(u_1 - \frac{s_1}{2})^2 - u_2^2][(u_2 - \frac{s_2}{2})^2 - u_1^2]} \end{aligned} \quad (\text{C.19})$$

This is the representation that we have used in (5.12).

References

- [1] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, *Commun. Math. Phys.* **147** (1992) 1.
- [2] E. Witten, *The N matrix model and gauged WZW models*, *Nucl. Phys.* **B 371** (1992) 191.
- [3] E. Witten, *Algebraic geometry associated with matrix models of two dimensional gravity*, in *Topological methods in modern mathematics*, Publish or Perish, Houston TX U.S.A. (1993).
- [4] E. Brézin and S. Hikami, *Vertices from replica in a random matrix theory*, *J. Phys.* **A 40** (2007) 13545 [[arXiv:0704.2044](#)].
- [5] E. Brézin and S. Hikami, *Intersection theory from duality and replica*, *Commun. Math. Phys.* online first [[arXiv:0708.2210](#)].
- [6] E. Brézin and S. Hikami, *Intersection numbers of Riemann surfaces from Gaussian matrix models*, *JHEP* **10** (2007) 096 [[arXiv:0709.3378](#)].
- [7] E. Brézin and S. Hikami, *Universal singularity at the closure of a gap in a random matrix theory*, *Phys. Rev.* **E 57** (1998) 4140 [[cond-mat/9804023](#)].
- [8] E. Brézin and S. Hikami, *Level spacing of random matrices in an external source*, *Phys. Rev.* **E 58** (1998) 7176 [[cond-mat/9804024](#)].
- [9] A. Okounkov, *Generating functions for intersection numbers on moduli spaces of curves*, *Int. Math. Res. Not.* **18** (2002) 933 [[math.AG/0101201](#)].
- [10] I.P. Goulden, J.L. Harer and D.M. Jackson, *A geometric parametrization for the virtual Euler characteristics of the moduli spaces of real and complex algebraic curves*, *Trans. Am. Math. Soc.* **353** (2001) 4405 [[math.AG/9902044](#)].

- [11] D.J. Gross and M.J. Newman, *Unitary and Hermitian matrices in an external field II: the Kontsevich model and continuum Virasoro constraints*, *Nucl. Phys. B* **380** (1992) 168 [[hep-th/9112069](#)].
- [12] E. Brézin and S. Hikami, *Characteristic polynomials of random matrices*, *Commun. Math. Phys.* **214** (2000) 111 [[math-ph/9910005](#)].
- [13] E. Brézin, S. Hikami and A.I. Larkin, *Level statistics inside the vortex of a superconductor and symplectic random matrix theory in an external source*, *Phys. Rev. B* **60** (1999) 3589 [[cond-mat/9902037](#)].
- [14] E. Brézin and S. Hikami, *Characteristic polynomials of real symmetric random matrices*, *Commun. Math. Phys.* **223** (2001) 363 [[math-ph/0103012](#)].
- [15] E. Brézin and S. Hikami, *New correlation functions for random matrices and integrals over supergroups*, *J. Phys. A* **36** (2003) 711 [[math-ph/0208001](#)].
- [16] Harish Chandra, *Invariant differential operators on a semisimple Lie algebra*, *Proc. Nat. Acad. Sci.* **42** (1956) 252.